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INTERNAL PROPERTIES OF SOME RHEOLOGICAL MODELS OF A VISCOELASTIC FLUID†

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The problem of the existence and non-existence of solutions for certain models of an incompressible viscoelastic fluid is considered. It is found that the Taylor–Couette and Hagen–Poiseuille flows have no standard azimuthal and one-dimensional solutions within the framework of the series of Oldroyd models of a viscoelastic fluid. It is found that in the critical case, i.e. at the border between the existence and non-existence of solutions certain characteristic features of Couette and Poiseuille flows become universal and do not depend on the parameters of the models.

1. TAYLOR–COUETTE FLOW

CONSIDER the flow of an incompressible viscoelastic fluid between two coaxal cylinders of radii R_1 and R_2 respectively, rotating with angular velocity ω_1 and ω_2. The fluid flow is described by the equations

$$\rho d\mathbf{v}/dt = \operatorname{div} \sigma - \nabla p, \quad \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

The stress tensor σ is connected with the deformation rate tensor D by the relations

$$\sigma + \lambda F_{abc}(\sigma) = 2\eta[D + \bar{\lambda} F_{\alpha\beta}(D)] \tag{1.2}$$

$$F_{abc}(\sigma) = d\sigma/dt - \Omega\sigma + \sigma\Omega - a(D\sigma + \sigma D) + b(\sigma : D)I + c(\operatorname{Tr}\sigma)D$$

$$F_{\alpha\beta}(D) = dD/dt - \Omega D + D\Omega - 2\alpha D^2 + \beta(D : D)I$$

$$D = {}^1/_2[\nabla\mathbf{v} + (\nabla\mathbf{v})^T], \quad \Omega = {}^1/_2[\nabla\mathbf{v} - (\nabla\mathbf{v})^T]$$

which defines the generalized, eight-parameter Oldroyd model [1]. Here λ is the relaxation time, $\bar{\lambda}$ is the delay time, η is the coefficient of viscosity, I denotes the unit tensor, a, b, c, α, β are the dimensionless parameters of the model, and D and Ω denote the symmetric and antisymmetric part of the tensor $\nabla\mathbf{v}$.

In the special case of $\bar{\lambda} = 0$ the Oldroyd model becomes the Maxwell model (the case when $b = c = 0$ and $a = 1$ corresponds to the supraconvective, and the case when $b = c = 0$ and $a = -1$ corresponds to the infraconvective Maxwell model). We also note that when $a = \alpha = 1$ and $b = c = \beta = 0$, the eight-parameter Oldroyd model reduces to the Oldroyd B-model.

We shall seek the azimuthal solution, which has the following form in a cylindrical system of coordinates r, θ, z:

$$\mathbf{v} = (0,\ v(r),\ 0), \quad p = p(r), \quad \sigma = \sigma(r) \tag{1.3}$$

In this case we obtain

$$D = \frac{1}{2} U \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad \Omega = \frac{1}{2}\omega \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}; \quad U = r\frac{d}{dr}\left(\frac{v}{r}\right), \quad \omega = \frac{1}{2}\frac{d}{dr}(vr)$$

Substituting expressions (1.3) into the equations of motions (1.2), we obtain

$$\frac{d}{dr}(r^2\sigma_{r\theta}) = 0, \quad \frac{1}{r}\left[\frac{d}{dr}(r\sigma_{rr}) - \sigma_{\theta\theta}\right] = \frac{dp}{rd} - \frac{\rho v^2}{r} \tag{1.4}$$

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The first equation of (1.4) yields $\sigma_{r\theta} = -2m/r^2$ where m is an arbitrary constant. Using the previous results we obtain from Eqs (1.2),

$$\sigma_{rr}=\lambda 2m/r^{-2}U(1-a+b)+\bar{\lambda}\eta U^2(1-\alpha+\beta)$$
$$\sigma_{\theta\theta}=-\lambda 2m/r^{-2}U(1+a-b)-\bar{\lambda}\eta U^2(1+\alpha-\beta) \tag{1.5}$$
$$\eta U=-2m/r^{-2}+{}^1/_2\lambda U[\sigma_{\theta\theta}(1-a+c)-\sigma_{rr}(1+a-c)]$$

Using the first two equations of (1.5) we can transform its last equation to the form

$$\eta\lambda\bar{\lambda}BU^3+2mr^{-2}\lambda^2AU^2+\eta U+2mr^{-2}=0 \tag{1.6}$$
$$A=1-(a-b)(a-c),\quad B=1-(a-c)(\alpha-\beta)$$

Let us now investigate some special cases. For the Jeffrey model ($\lambda = 0$) the azimuthal velocity is identical with the classical expression for a Newtonian fluid. For the Maxwell model ($\bar{\lambda} = 0$) Eq. (1.6) becomes quadratic and we can obtain from its solution the final expression for the azimuthal component of flow velocity

$$v=r\left(\omega_1+\int_{R_1}^{r}U(x)\frac{dx}{x}\right),\quad U(r)=\frac{-\eta r^2+\sqrt{\eta^2r^4-16m^2\lambda^2A}}{4m\lambda^2A} \tag{1.7}$$

We note that the boundary condition of adhesion on the inner cylinder holds, and the requirement that the boundary condition should hold on the outer cylinder determines the constant m.

The solution (1.7) exists only in the case when the expression under the radical sign is non-negative. Let us study in more detail the case of an isolated rotating cylinder, $R_2 \to \infty$. Integrating relation (1.7) we obtain (De is the Deborah number)

$$v=r\omega_1\left[1+\frac{f(R_1)-f(r)}{\mathrm{De}\sqrt{A}}\right];\qquad f(r)=\left(\frac{r}{R}\right)^2-\Phi(r)+\operatorname{arctg}\Phi(r)$$
$$\Phi(r)=\sqrt{\left(\frac{r}{R}\right)^4-1},\quad R=\left(\frac{4m\lambda}{\eta}\right)^{1/2}A^{1/4},\quad \mathrm{De}=2\omega_1\lambda \tag{1.8}$$

The obvious boundary condition $v = 0$ and $r = \infty$ yields the relation

$$\mathrm{De}\sqrt{A}=\pi/2-f(R_1) \tag{1.9}$$

which defines implicitly the constant m as a function of the Deborah number, the Reynolds number $\mathrm{Re} = \rho\omega_1 R_1^2/\eta$ and the parameter A. Note that the function $f(r)$ is defined for $r \geqslant R$ and increases from 1 to $\pi/2$ as r increases from R to infinity. Thus from (1.9) we find that the standard solution of type (1.3) exists for the Maxwellian fluid within the range of Deborah numbers:

$$0\leqslant\mathrm{De}\leqslant(\pi/2-1)A^{-1/2} \tag{1.10}$$

For small values of the Deborah numbers, the moment acting on the cylinder is given by the following expression:

$$M=-2\pi r^2\sigma_{r\theta}=4\pi m=4\pi\eta\omega_1R_1^2\left[1-\frac{1}{3}A\,\mathrm{De}^2+\frac{7}{30}(A\,\mathrm{De}^2)^2+\ldots\right] \tag{1.11}$$

We note that when the De number increases the dimensionless moment $\mu = M/4\pi\eta\omega_1 R_1^2$ decreases from unity (for a Newtonian fluid) to $(\pi-2)^{-1} \approx 0.876$ on the upper limit of the boundary of existence. The drop in the value of the torsional moment can be seen directly from the asymptotic expansion of (1.11) for small De numbers; when the De numbers are arbitrary, this follows from an elementary analysis of the condition for (1.9) to be solvable.

In the previous results we assumed implicitly that the parameter A is non-negative, and this indeed is the case in most commonly used rheological models [1]. For completeness a similar analysis was carried out for the case when $A<0$. It was found that in this case a solution exists everywhere and the torsional moment increases as the value of the Deborah number increases.

Thus we have shown that azimuthal Couette flow exists in the case when $A>0$, over a limited range of Deborah numbers.

2. POISEUILLE FLOW

Let us consider a steady, unidirectional flow of viscoelastic fluid in a pipe of arbitrary cross-section, induced by a pressure gradient. We have for such a flow

$$\mathbf{v} = (0, 0, w(x, y)), \quad D = \frac{1}{2}\begin{Vmatrix} 0 & 0 & w_x \\ 0 & 0 & w_y \\ w_x & w_y & 0 \end{Vmatrix}, \quad \Omega = \frac{1}{2}\begin{Vmatrix} 0 & 0 & -w_x \\ 0 & 0 & -w_y \\ w_x & w_y & 0 \end{Vmatrix} \tag{2.1}$$

Here x, y, z are Cartesian coordinates (the z axis is directed along the pipe axis), $w_x = \partial w/\partial x$ and $w_y = \partial w/\partial y$. Expressions (2.1) suggest that the stress tensor should be sought in the similar form

$$\sigma = \begin{Vmatrix} 0 & 0 & \sigma_1 \\ 0 & 0 & \sigma_2 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{Vmatrix}$$

After substituting these expressions into the last two relations of (1.2) we see that the "derivatives" $F_{abc}(\sigma)$ and $F_{\alpha\beta}(D)$ have the required form, provided that the following conditions hold:

$$a=1, \quad b=0, \quad \alpha=1, \quad \beta=0 \tag{2.2}$$

Note that the Oldroyd model with constraints (2.2) is a one-parameter generalization of the Oldroyd B-model (the latter corresponds to the case $c = 0$).

Henceforth, we shall consider precisely the Oldroyd model with constraints (2.2). In this case we find, for the equations of state,

$$\sigma_{1,2}=\eta j w_{x,y}, \quad \sigma_3=2\eta(\lambda-\bar{\lambda})(\nabla w)^2/[1+c\lambda^2(\nabla w)^2]$$
$$j = [1+c\lambda\bar{\lambda}(\nabla w)^2]/[1+c\lambda^2(\nabla w)^2] \tag{2.3}$$

Substituting expressions (2.3) into the equation of motion (1.1) we obtain

$$\frac{\partial}{\partial x}(j w_x) + \frac{\partial}{\partial y}(j w_y) = \frac{1}{\eta}\frac{dp}{dz} \tag{2.4}$$

Note that (2.4) can be interpreted as the equation of continuity for potential flows of an ideal compressible fictitious gas with sources. Here j will be regarded as the density of the fictitious gas. The longitudinal velocity $w(x, y)$ of flow of a viscoelastic fluid serves as the potential, so that $\mathbf{q} = \nabla w = (w_x, w_y)$ is the velocity of the fictitious gas; the constant pressure gradient is then regarded as the source intensity. This curious mathematical analogy between unidirectional viscoelastic incompressible flows and compressible flows of a fictitious gas is found to be useful in a qualitative analysis of the properties of a viscoelastic fluid.

Let us now analyse the problem of the existence of solutions of Eq. (2.4). For the Oldroyd B-model the parameter $c = 0$, so that the density of the fictitious gas is constant and Eq. (2.4) reduces to Poisson's equation. It follows that the unidirectional viscoelastic flows of the Oldroyd B-fluid are described by the same equation as the unidirectional flows of a classical field. A solution may not exist only when $c\neq 0$, more more accurately when $c>0$. In this case Eq. (2.4) takes the following form for the Maxwellian fluid ($\bar{\lambda} = 0$):

$$\nabla\cdot(\mathbf{q}/(1+c\lambda^2q^2))=\eta^{-1}dp/dz \tag{2.5}$$

Integrating Eq. (2.5) over the pipe cross-section area S, we obtain

$$\oint \frac{\mathbf{q}\cdot\mathbf{n}}{1+c\lambda^2q^2}\,dl = \frac{S}{\eta}\frac{dp}{dz}$$

where n is the unit vector of the outer normal. Using the inequality

$$q/(1+c\lambda^2q^2)\leqslant 1/(2\sqrt{c\lambda^2})$$

and denoting by L the perimeter of the pipe cross-section, we arrive at the necessary condition for a solution to exist

$$0\leqslant \mathrm{De}\leqslant 1/\sqrt{c}, \quad \mathrm{De}=2SL^{-1}\lambda(-\eta^{-1}dp/dz) \tag{2.6}$$

We note that the condition for a unidirectional Poiseuille flow (2.6) to exist resembles the condition for Couette flow (1.10) to exist. We also note that when the constraints (2.2) imposed on the parameters of the Oldroyd model are taken into account, the constants A and c become identical.

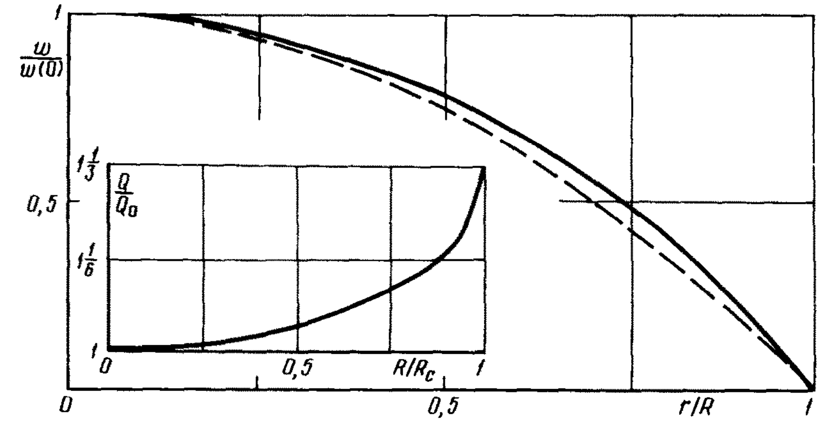

FIG. 1.

In the case of a circular pipe the necessary condition for solution (2.6) to exist also becomes sufficient. This follows from the explicit solution of the initial equation (2.4)

$$q = \frac{1}{\sqrt{c\lambda^2}}\frac{R_c}{r}[F(r)-1], \quad R_c = \left(-\frac{1}{\eta}\frac{dp}{dz}\sqrt{c\lambda^2}\right)^{-1}, \quad F(r) = \sqrt{1-\left(\frac{r}{R_c}\right)^2} \tag{2.7}$$

We see that the solution exists only for pipes of relatively small radius $R (R \leq R_c)$, in accordance with condition (2.6). It is interesting to note that the viscoelastic Poiseuille flow in a circular pipe of critical radius $R = R_c$ corresponds to the flow of a fictitious gas whose flow velocity at the pipe walls is found to be equal to the velocity of sound $q = -(c\lambda^2)^{-1/2}$. The non-existence of a solution in the present problem in terms of the fictitious gas, resembles the pipe blocking effect known in gas dynamics.

The longitudinal flow velocity of a viscoelastic Maxwellian fluid can be found directly from (2.7)

$$w(r) = \frac{R_c}{\sqrt{c\lambda^2}}\left[F(r)-F(R)-\ln\frac{1+F(r)}{1+F(R)}\right]$$

A graph depicting the relationship $w = w(r)$ at $R/R_c = 0.25$ (the solid line) and $R/R_c = 1$ (the dashed line) is given in Fig. 1.

The flow rate through the pipe is given by the expression

$$Q = \int_0^R w2\pi r\,dr = \frac{\pi}{6}\frac{R_c^3}{\sqrt{c\lambda^2}}[1-F(R)]^2[1+2F(R)]$$

The figure also shows the relation between the ratio Q/Q_0 and the dimensionless radius of the pipe R/R_c

$$Q/Q_0 = {}^4/_3[1+2F(R)]/[1+F(R)]^2$$

where Q_0 is the flow rate in the case of a Newtonian fluid. We see that the flow rate of a Newtonian fluid is always smaller than that of a Maxwellian fluid. When $R = R_c$, the ratio Q/Q_0 becomes independent of the parameters of the model and is equal to $4/3$.

We note that when $c<0$, the solution of the Poiseuille problem always exists, and has the form

$$w(r) = \frac{R_c}{\sqrt{-c\lambda^2}}\ln\frac{r}{R} + \frac{1}{2}\ln\left[\frac{G(R)+1}{G(R)-1}\frac{G(r)-1}{G(r)+1}\right], \quad G(r) = \sqrt{1+\left(\frac{r}{R_c}\right)}$$

Thus the existence or non-existence of the solution of the Poiseuille problem depends decisively on the sign of the parameter c, just as on the sign of the parameter A in the Couette problem. We also note that similar results can also be obtained for Poiseuille flow in a slot. In particular, the necessary condition of existence again has the form (2.6), with the Deborah number given by the relation

$$De = H\lambda(-\eta^{-1}dp/dz)$$

where H is the width of the channel. We find that in this case the flow rate Q also exceeds the classical value of Q for a Newtonian fluid. For a channel of critical width $H = R_c$ the ratio Q/Q_0 is also universal and equal to $6(1-\pi/4) = 1.288$.

In conclusion we note that the problems of the existence of solutions of equations of viscoelastic fluid for various types of flow has recently been given a considerable amount of attention (see e.g. [2–5]).

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